Journal of Approximation Theory **116**, 126–140 (2002)
doi:10.1006/jath.2001.3664, available online at http://www.idealibrary.com on IDEAL®

Approximation of Convex Functions on the Dual of Banach Spaces

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Communicated by Günther Nürnberger

Received April 19, 2001; accepted in revised form October 18, 2001

This paper shows that every w^*-lower semicontinuous Lipschitzian convex function on the dual of a locally uniformly convexifiable Banach space, in particular, the dual of a separable Banach space, can be uniformly approximated by a generically Fréchet differentiable w^*-lower semicontinuous monotone-nondecreasing Lipschitzian convex function sequence

Key Words: convex function; Fréchet differentiability; approximation; Banach space.

1. INTRODUCTION

A lot of articles have shown that approximation methods using various kinds of smoothness of Banach spaces to establish smooth variational principles and to resolve problems concerning Banach space theory are very powerful (see, for instance, [2, 6, 7, 9, 13, 17, 18]). The authors in these papers mainly applied norms or bump functions (i.e., continuous functions defined whole spaces with nonempty bounded supports) with some kinds of smoothness to construct suitable procedures in approximation for obtaining desired results. People also often consider approximation of convex functions and its applications, such as "approximating a lower semicontinuous proper convex funtion by its inf-convolution sequence," "approximating to a norm by a smooth norm sequence," and "approximating to a continuous convex function by smooth convex functions," etc. (see, for instance, [4; 7, p.90, Problem II. 5; 10–12; 19]). This paper focuses on the question of w^*-lower semicontinuous convex functions approximated by generically Fréchet differentiable convex functions on dual spaces and mainly shows the following result.

THEOREM. *Suppose that E is a Banach space admitting an equivalent locally uniformly convex norm (in particular, a separable space). Then for*

0021-9045/02 $35.00

every extended-real-valued w^*-lower semicontinuous proper convex function f on E^*, there exists a sequence of w^*-lower semicontinuous convex functions $\{f_n\}$ on E^* such that

(i) *For each integer $n \geqslant 1$, f_n is Lipschitzian and generically Fréchet differentiable;*

(ii) *$\{f_n\}$ is monotone non-decreasing and dominated by f, that is,*

$$f_n(x^*) \leqslant f_{n+1}(x^*) \leqslant f(x^*), \qquad \text{for all} \quad n \geqslant 1 \quad \text{and} \quad x^* \in E^*;$$

(iii) $\lim_{n \to \infty} f_n(x^*) = f(x^*)$ *for all $x^* \in E^*$.*

If, in addition, f is bounded on each bounded subset of E^*, then

(iv) *$f_n \to f$ uniformly on each bounded subset of E^*.*

A convex function on a Banach space E is called generically Fréchet differentiable if it is Fréchet differentiable at each point of a dense G_δ subset of E [9]. The space E is said to be an Asplund space provided every continuous convex function on E is generically Fréchet differentiable [15] (see also [16]); and the dual E^* of E is said to be a w^*-Asplund space if every w^*-lower semicontinuous and norm continuous convex function on E^* is generically Fréchet differentiable [5] (see also [1]).

We should remark that Collier [5] showed E^* is a w^*-Asplund space if and only if the predual E has the Radon–Nikodým property (RNP). The main theorem presented in this paper further explains that though a locally uniformly convexifiable Banach space E would not have the RNP, it is very "near" to having the RNP. This paper consists of five sections. The second section lists some definitions and properties which will be used in the sequel; the third section presents some versions for approximants to Minkowski functionals; the fourth section shows the main theorem in uniform approximation sense; and the last section gives some remarks on pointwise approximation and on differentiability of convex functions.

2. PRELIMINARIES

We will always assume that $(E, \|\cdot\|)$ is a real Banach space and $(E, \|\cdot\|)^*$ its dual, we also simply denote them by E and E^*, resp., if no confusion is caused. By $B_{|\cdot|}$ we mean the closed unit ball of $(E, |\cdot|)$ for an equivalent norm $|\cdot|$ on E, and by B the closed unit ball determinated by the original norm $\|\cdot\|$ on E. For a set A in E, $\bar{A}$ and $\bar{A}^{w^*}$ stand for the norm and the w^*-closure of A, resp.; if E is not a dual space, then $\bar{A}^{w^*}$ denotes the w^*-closure of the canonical embedding of A in the bidual E^{**}.

Convex Functions, Conjugates, and Subdifferential Mappings

An extended real-valued convex function f on E is said to be proper if it is nowhere $-\infty$ valued and its effective domain $dom\, f \equiv \{x \in E : f(x) < \infty\} \neq \emptyset$. We denote the epigraph of the function f by $epi\, f \equiv \{(x, r) \in E \times R : f(x) \leqslant r\}$. The conjugate f^* of f on E^* and the biconjugate f^{**} on E are defined by

$$f^*(x^*) = sup\{\langle x^*, x\rangle - f(x) : x \in E\}, \qquad x^* \in E^*$$

and

$$f^{**}(x) = sup\{\langle x^*, x\rangle - f^*(x^*) : x^* \in E^*\}, \qquad x \in E,$$

respectively. Clearly, f^* is always w^*-lower semicontiuous on E^*. The subdifferential mapping ∂f of f on E is defined by

$$\partial f(x) = \{x^* \in E^* : f(y) - f(x) \geqslant \langle x^*, y - x\rangle \textit{ for all } y \in E\}, \qquad x \in E.$$

We can easily observe

$$\partial f(x) = \{x^* \in E^* : f(x) + f^*(x^*) = \langle x^*, x\rangle\}.$$

PROPOSITION 2.1. *Suppose that f is a continuous convex function on a nonempty open convex set D of E. Then the subdifferential mapping ∂f is locally bounded and norm-to-w^* upper semicontinuous at every point of D.*

PROPOSITION 2.2 [4]. *Suppose that f is a lower semicontinuous proper convex function on E. Then $\partial f(E)$ (the image of the subdifferential mapping ∂f) is dense in dom f^*.*

Convex Sets, Support Functions, and Minkowski Functionals

For a nonempty subset $A \subset E(E^*)$, σ_A stands for the support function of A. Clearly, $\sigma_A = \sigma_{\overline{co}\, A}$ ($\sigma_{\overline{co}^{w^*}(A)}$) is always an extended-real-valued w^*-lower semicontinuous (lower semicontinuous) proper sublinear on $E^*(E)$. Conversely, for each w^*-lower semicontinuous (lower semicontinuous) proper sublinear functional p on $E^*(E)$, there exists a unique closed (w^*-closed) convex set C in $E(E^*)$ such that $p = \sigma_C$.

An extended-real-valued function p on E is said to be a Minkowski functional if there exists a convex set C with $0 \in C$ such that $p(x) = inf\{\lambda \geqslant 0 : x \in \lambda C\}$, $x \in E$; In this case, we say p is generated by C. p is lower semicontinuous (continuous) if and only if C is closed ($0 \in int\, C$, resp.); and we have $C = \{x \in E : p(x) \leqslant 1\}$. For such a convex set C, $C^* \equiv \{x^* \in E^* : \langle x^*, x\rangle \leqslant 1, \forall x \in C\}$ is termed the polar of C.

Proposition 2.3. *Suppose that p is a lower semicontinuous Minkowski functional generated by C. Then*

(i) $p(x)=\sigma_{C^*}(x)$ *for all* $x\in E$;

(ii) $\partial p(E)=\partial p(0)=C^*$ (*thus,* $x^*\in\partial p(x)\Leftrightarrow x^*\in C^*$ *with* $\langle x^*, x\rangle = p(x)$);

(iii) $C^{**}\equiv \bar{C}^{w^*}=\partial\sigma_C(E^*)=\partial\sigma_C(0)$.

(w^*-) Slices and (w^*-) Strongly Exposed Points of Convex Sets

For a nonempty bounded subset A of $E(E^*)$, $x^*\in E^*\backslash\{\theta\}$ $(x\in E\backslash\{\theta\})$, $\alpha>0$, we say

$$S(A, x^*, \alpha)\equiv\{x\in A : \langle x^*, x\rangle > \sigma_A(x^*)-\alpha\}$$

$$(S(A, x, \alpha)\equiv\{x^*\in A : \langle x^*, x\rangle > \sigma_A(x)-\alpha\})$$

a slice (w^*-slice) of A.

Assume that $C\subset E(E^*)$ is a bounded closed (w^*-closed) convex set. A point $x\in C(x^*\in C)$ is said to be a strongly (w^*-strongly) exposed point of C if there exists a functional $x^*\in E^*(x\in E)$ such that $\{S(C, x^*, \alpha)\}_{\alpha>0}$ $(\{S(C, x, \alpha)\}_{\alpha>0})$ forms a local base of the point $x(x^*)$ of C; this means that for every sequence $\{x_n\}(\{x_n^*\})$ in C, $\langle x^*, x_n\rangle\to\sigma_C(x^*)$ $(\langle x_n^*, x\rangle\to\sigma_C(x))$ implies $x_n\to x$ $(x_n^*\to x^*)$; In this case, we call the functional $x^*(x)$ is a strongly (w^*-strongly) exposing functional of C and strongly (w^*-strongly) exposing C at $x(x^*)$, or we simply say $x^*(x)$ is a strongly (w^*-strongly) exposing functional of $x(x^*)$.

Proposition 2.4. *Suppose that C is a bounded closed set in E and $C^{**}\equiv\bar{C}^{w^*}$. If $x\in C$ is a strongly exposed point of C and strongly exposed by $x^*\in E^*$, then x is a w^*-strongly exposed point of C^{**} and w^*-strongly exposed by the same functional x^*.*

Proof. By hypothesis, $\{S(C, x^*, \alpha)\}_{\alpha>0}$ forms a local base of $x\in C$. Note $S(C, x^*, \alpha)\subset S(C^{**}, x^*, \alpha)\subset\bar{S}^{w^*}(C, x^*, \alpha)$ for every $\alpha>0$, and $\lim_{\alpha\to 0}$ *diam* $S(C, x^*, \alpha)=0$. w^*-lower semicontinuity of the bidual norm on E^{**} implies that

$$\textit{diam}\, S(C^{**}, x^*, \alpha)=\textit{diam}\, S(C, x^*, \alpha)\to 0 \qquad (\text{as } \alpha\to 0^+)$$

Fréchet Differentiability of Convex Functions

A real-valued convex function f on a nonempty open convex subset D of E is said to be Fréchet differentiable at x if there exists a (unique) $x^*\in E^*$ such that for every $\epsilon>0$, there is $\delta>0$,

$$0\leqslant f(x+y)-f(x)-\langle x^*, y\rangle\leqslant\epsilon\|y\|, \qquad \text{whenever} \quad y\in E \text{ with } \|y\|<\delta.$$

The function f is said to be generically Fréchet differentiable in D if it is everywhere Fréchet differentiable in a dense G_δ subset of D, or equivalently, f is densely Fréchet differentiable in D since the Fréchet differentiability point set of a continuous convex function is always a G_δ-subset.

PROPOSITION 2.5 [16, Proposition 5.11]. *Suppose that p is a continuous Minkowski functional on E with $\partial p(0) \equiv C^*$. Then p is Fréchet differentiable at x with the Fréchet derivative x^* if and only if x^* is a w^*-strongly exposed point of C^* and w^*-strongly exposed by x.*

PROPOSITION 2.6 [3]. *Suppose that f is a continuous convex function on E with $f(0) = -1$, and suppose that p is the Minkowski function generated by epi f. Then f is Fréchet differentiale at x with the Fréchet derivative x^* if and only if p is Fréchet differentiable at (x, r) and with the Fréchet derivative $r^*(x^*, -1)$ where $r = f(x)$ and $r^* = f^*(x^*)^{-1}$.*

Approximants to a Convex Function by Its Inf-Convolutions

An extended real-valued function f on a nonempty open subset D is said to be locally Lipschitzian around $x \in D$ if there exist an open neighborhood U of x and a constant L such that $|f(x) - f(y)| \leqslant L \|x - y\|$ whenever $x, y \in D$; and to be locally Lipschitzian on D if it is locally Lipschitzian around each point x of D. We denote by $L_D(f)$ the smallest constant such that $|f(x) - f(y)| \leqslant L_D(f) \|x - y\|$ whenever $x, y \in D$. The functions with $L_D(f) < \infty$ are termed Lipschitzian on D; If $D = E$ and $L_A(f) < \infty$ for each bounded subset A of E, then f is called b-Lipschitzian on E.

For an extended-real-valued proper convex function f on E, and for $n = 1, 2, \ldots$, the inf-convolutions f_n are defined by $f_n(x) = inf\{f(y) + n \|x - y\| : y \in E\}$. The following properties are classical (see, for instance, [20]).

PROPOSITION 2.7. *With the functions f and f_n as above, then*

(i) *$f_n \leqslant f_{n+1} \leqslant f$ for each integer $n \geqslant 1$;*

(ii) *f_n is Lipschitzian with $L_E(f_n) \leqslant n$ for all sufficiently large $n \geqslant 1$;*

(iii) *$\lim_{n \to \infty} f_n(x) = f(x)$ for all $x \in E$;*

(iv) *if f is locally Lipschitzian around $x \in E$, then there exist $n \geqslant 1$ and a neighborhood U of x such that $f_n = f$ in U.*

3. APPROXIMATION VERSIONS FOR MINKOWSKI FUNCTIONALS

A Minkowski functional p on a Banach space $(E, \|\cdot\|)$ is called an equivalent asymmetric norm (or simply, an a-norm) if there exist $a, b > 0$ such that

$$a\,\|x\| \leqslant p(x) \leqslant b\,\|x\| \qquad \text{for all} \quad x \in E.$$

Letting $p^* = \sigma_C$, where $C = \{x \in E : p(x) \leqslant 1\}$, then p^* is a w^*-lower semicontinuous a-norm on E^* with

$$b^{-1}\,\|x^*\| \leqslant p^*(x^*) \leqslant a^{-1}\,\|x^*\| \qquad \text{for all} \quad x^* \in E^*,$$

which is called the dual of the a-norm p.

We denote by (Ω, d) all continuous Minkowski functionals on E endowed with the metric d defined for $p, q \in \Omega$ by

$$d(p, q) = sup\{|p(x) - q(x)| : x \in B\}.$$

Clearly, (Ω, d) is a complete metric space. Let Ω_0 be the collection of all a-norms on E. Then we observe that Ω_0 is a dense open subset of (Ω, d); therefore (Ω_0, d) is itself a Baire space.

Dually, letting (Ω^*, d^*) denote all w^*-lower semicontinuous and norm continuous Minkowski functionals on E^* equipped with the metric d^* defined for $p^*, q^* \in \Omega^*$ by

$$d^*(p^*, q^*) = sup\{|p^*(x^*) - q^*(x^*)| : x^* \in B^*\}$$

and Ω_0^* denote the collection of all dual a-norms on E^*, then Ω_0^* is a dense open subset of the complete metric space (Ω_0^*, d^*). Without any difficulty to show that the mapping Φ: $(\Omega_0, d) \to (\Omega_0^*, d^*)$ defined by

$$\Phi(p) = p^*(\text{the dual of } p), \qquad p \in \Omega_0$$

is a homeomorphism from (Ω_0, d) onto (Ω_0^*, d^*). An a-norm p on the space $(E, \|\cdot\|)$ is said to be locally uniformly convex, if for any $x, x_n \in E$ with $p(x) = 1 = p(x_n)$ for $n = 1, 2, \cdots$, $p(x + x_n) \to 2$ implies $x_n \to x$, which is equivalent to that for any $x(\neq 0)$, $x_n \in E$, $p(x) + p(x_n) - p(x + x_n) \to 0 (n \to \infty)$ implies $x_n \to \lambda x$ for some $\lambda \geqslant 0$.

LEMMA 3.1. *Suppose that p is a locally uniformly convex a-norm and q is an extended-real-valued lower semicontinuous Minkowski functional on E. Let $r = p + q$ and $r^* = \sigma_C$, where $C = \{x \in E : r(x) \leqslant 1\}$. Then r^* is generically Fréchet differentiable in E^*.*

Proof. Noting that $C(p) \equiv \{x \in E : p(x) \leqslant 1\}$ is closed bounded convex and $C \subset C(p)$, lower semicontinuity of r implies that C is a closed bounded convex set with $0 \in C$. Therefore $r^* \equiv \sigma_C$ is a w^*-lower semicontinuous and norm continuous Minkowski functional on E^*, that is, $r^* \in \Omega^*$. By the Bishop–Phelps theorem (see, for instance [16, pp. 51–52]), the set G of all support functionals of C are dense in E^*. It suffices to show that r^* is Fréchet differentiable at each point of G.

Let $S = \{x^* \in E^* : r^*(x^*) > 0\}$ and $K = E^* \setminus S$. Note $S \cup \mathit{int}\, K$ is dense in E^*, S is an open set of E^* and $r^* = \sigma_C$ is Fréchet differentiable at each point of $\mathit{int}\, K$. We need only proving that f is Fréchet differentiable at each point of $G \cap S$. Given $x^* \in G \cap S$, by definition we can assume that $\langle x^*, x \rangle = \sigma_C(x^*) = 1$ for some point $x \in C$. So that $x^* \in C^*$(the polar of C).

Noting that $x \in \partial \sigma_C(x^*)$ and noting Proposition 2.3(iii), by Proposition 2.5, it suffices to verify that $x \in C^{**}(\equiv \bar{C}^{w^*} = \partial \sigma_C(0))$ is a w^*-strongly exposed point of C^{**} and is w^*-strongly exposed by x^*.

Suppose, to the contrary, that $x \in C^{**}$ is not w^*-strongly exposed by x^*. Then, by Proposition 2.4, x is not a strongly exposed point of C strongly exposed by x^*. Therefore there exist a sequence $\{x_n\}$ in C and $\epsilon > 0$ such that $\langle x^*, x_n \rangle \to \sigma_C(x^*) = 1$ and $\|x - x_n\| \geqslant \epsilon$ for all $n \geqslant 1$. Noting $\sigma_{C^*} = r$ which is less or equal to 1 at each point of C, we have

$$2 \geqslant r(x) + r(x_n) \geqslant r(x + x_n) = \sigma_{C^*}(x + x_n) \geqslant \langle x^*, x + x_n \rangle \to 2.$$

On the other hand,

$$\begin{aligned} r(x+x_n) &\equiv p(x+x_n) + q(x+x_n) \\ &\leqslant p(x) + p(x_n) + q(x) + q(x_n) \\ &= r(x) + r(x_n). \end{aligned}$$

Combining with the inequalities above we observe

$$p(x) + p(x_n) - p(x + x_n) \to 0$$

and

$$q(x) + q(x_n) - q(x + x_n) \to 0$$

and locally uniform convexity of p tells us $x_n \to \lambda x$ for some $\lambda \geqslant 0$. This and $\langle x^*, x_n \rangle \to \langle x^*, x \rangle = 1$ further say that $\lambda = 1$, and this is a contradiction which completes our proof.

COROLLARY 3.2. *The dual norm of an equivalent locally uniformly convex norm on a Banach space E is generically Fréchet differentiable in E^*.*

THEOREM 3.3. *Suppose that* $(E, \|\cdot\|)$ *admits an equivalent locally uniformly convex norm* $|\cdot|$. *Then for every* w^**-lower semicontinuous and norm continuous Minkowski functional* p^* *on* E^* *there exists a sequence of* w^**-lower semicontinuous and norm continuous Minkowski functionals* $\{p_n^*\}$ *on* E^* *such that*

(i) $p_n^* \leqslant p_{n+1}^* \leqslant p^*$ *for all integers* $n \geqslant 1$;
(ii) *for each* $n \geqslant 1$, p_n^* *is generically Fréchet differentiable;*
(iii) $p_n^* \to p^*$ *uniformly on each bounded subset of* E^*.

Proof. Suppose that p^* is a w^*-lower semicontinuous and norm continuous Minkowski functional on E^*, and $|\cdot|$ is an equivalent locally uniformly convex norm on E. Let

$$C^* = \{x^* \in E^* : p^*(x^*) \leqslant 1\}$$

and

$$C = \{x \in E : \langle x^*, x \rangle \leqslant 1 \text{ for all } x^* \in C^*\}.$$

Then $\sigma_C = p^*$ and C is bounded since p^* is continuous. Boundedness of C and positive homogeneity allow us to assume $C \subset B_{|\cdot|}$. Let $p = \sigma_{C^*}$. Then p is an extended-real-valued lower semicontinuous proper Minkowski functional on E.

For each $n \geqslant 1$, let $p_n = \frac{1}{n}|\cdot| + p$ and $C_n = \{x \in E : p_n(x) \leqslant 1\}$. Then $C_n \subset C_{n+1} \subset C$ and C_n are closed since p_n are also lower semicontinuous Minkowski functionals on E for all $n \geqslant 1$.

Noting $\frac{1}{n}|\cdot|$ are equivalent locally uniformly convex norms on E for all $n \geqslant 1$, it follows from Lemma 3.1 that $\sigma_{C_n} \equiv p_n^* \leqslant p_{n+1}^*$ are generically Fréchet differentiable in E^*. Thus (i) and (ii) are proved.

To show (iii). Noting for every $n \geqslant 1$,

$$\begin{aligned} B_{|\cdot|} \supset C \supset C_n &\equiv \{x \in E : p_n(x) \leqslant 1\} \\ &= \left\{x \in E : p(x) \leqslant 1 - \frac{1}{n}|x|\right\} \\ &\supset \left\{x \in E : p(x) \leqslant 1 - \frac{1}{n}\right\} \\ &= \left(1 - \frac{1}{n}\right) C. \end{aligned}$$

So we have $p^* = \sigma_C \geqslant \sigma_{C_n} = p_n^* \geqslant (1 - \frac{1}{n})\, p^*$, and this explains (iii) holds.

THEOREM 3.4. *Suppose that the space E admits an equivalent locally uniformly convex norm. Then w^*-lower semicontinuous and norm continuous Minkowski functionals contains a dense G_δ subset of (Ω^*, d^*).*

Proof. Since Ω_0^* and Ω_0 are dense open subsets of the complete metric spaces (Ω^*, d^*) and (Ω, d), resp., and since the mapping $\Phi: p \to p^*$ is a homeomorphism from (Ω_0, d) onto (Ω_0^*, d^*), it suffices to show that locally uniformly convex a-norms form a dense G_δ-subset of (Ω_0, d). It is contained in [7, proof of Theorem 4.1.i, pp. 54–55] but with the one change —substituting the a-norms for the norms on E.

COROLLARY 3.5. *Suppose that E is a separable space. Then*

(i) *For every w^*-lower semicontinuous and norm continuous Minkowski functional p^* on E^* there exists a w^*-lower semicontinuous and norm continuous generically Fréchet differentiable monotone non-decreasing Minkowski functional sequence $\{p_n^*\}$ which uniformly converges to p^* on each bounded subset of E^*;*

(ii) *Generically Fréchet differentiable elements in (Ω^*, d^*) contain a dense G_δ-subset of (Ω^*, d^*).*

Proof. By Theorems 3.3 and 3.4, it is trivial since every separable Banach space is locally uniformly convexifiable (see, for instance, [7, 8]).

4. UNIFORM APPROXIMANTS TO CONVEX FUNCTIONS

In this section, we deal with continuous convex functions.

LEMMA 4.1. *Suppose that f is a continuous convex function on a nonempty open convex set D of E. Then we have*

$$L_D(f) = \sup\{\|x^*\|: x^* \in \partial f(D)\},$$

where $L_D(f)$ denotes the Lipschitz norm of f on D.

Proof. For any $x \in D$, $x^* \in \partial f(x)$, and $\epsilon > 0$, let $z \in E$ with $\|z\| = 1$ such that $\langle x^*, z\rangle > \|x^*\| - \epsilon$. Openness of D says there is $\delta > 0$ such that $x + y \in D$ whenever $\|y\| \leqslant \delta$. Thus, by definition

$$f(x+\delta z) - f(x) \geqslant \langle x^*, \delta z\rangle > \delta(\|x^*\| - \epsilon).$$

Arbitrariness of x^* and ϵ explain that

$$L_D(f) \geqslant sup\{\|x^*\|: x^* \in \partial f(D)\}.$$

On the other hand, $\forall x, y \in D$, we can assume $f(y) \geqslant f(x)$. Therefore

$$|f(y)-f(x)| = f(y)-f(x) \leqslant \langle y^*, y-x\rangle \leqslant \|y^*\| \, \|y-x\|, \qquad \forall y^* \in \partial f(y),$$

and this implies $L_D(f) \leqslant \sup\{\|x^*\|: x^* \in \partial f(D)\}$.

PROPOSITION 4.2. *Suppose that f is a continuous convex function on E. Then f is b-Lipschitzian on E if and only if f is bounded on each bounded subset of E.*

Proof. It suffices to show sufficiency. Suppose that f is not Lipschitzian on some open convex bounded set D. Then, by Lemma 4.1, there exist sequences $\{x_n\}$ in D and $\{x_n^*\}$ in $\partial f(D)$ with $x_n^* \in \partial f(x_n)$ for all n such that $\|x_n^*\| \to \infty$. Let $\{z_n\}$ in E with $\|z_n\| = 1$ for all n such that $\langle x_n^*, z_n\rangle > \|x_n^*\| - \frac{1}{n}$. Noting both $\{x_n + z_n\}$ and $\{x_n\}$ are bounded sequences, there exists $M > 0$ such that

$$M \geqslant f(x_n + z_n) - f(x_n) \geqslant \langle x_n^*, z_n\rangle > \|x_n^*\| - \frac{1}{n} \to \infty,$$

and this is a contradiction.

COROLLARY 4.3. *Suppose that f and g are continuous convex functions on E with $g \leqslant f$. If f is b-Lipschitzian on E, then g is also b-Lipschitzian.*

Proof. It suffices to note every continuous convex function is bounded below on each bounded set and note that b-Lipschitz of f implies g is bounded above on each bounded set since $f \geqslant g$.

LEMMA 4.4. *Suppose that f and g are continuous convex functions on the Banach space E with $f \geqslant g$. Then $L_E(f) \geqslant L_E(g)$.*

Proof. By Proposition 2.2, $\partial f(E)$ and $\partial g(E)$ are dense in $dom\ f^*$ and $dom\ g^*$, resp. Noting $dom\ f^* \supset dom\ g^*$ whenever $f \geqslant g$, and it immediately follows from Lemma 4.1.

LEMMA 4.5. *Suppose that f and f_n $(n = 1, 2, \ldots)$ are Lipschitzian convex functions on a Banach space E with $f(0) < 0$ and with $f_n \leqslant f$ for all integers $n \geqslant 1$, and suppose that p and p_n are the Minkowski functionals on $E \times R$ generated by the corresponding epigraphs $epi\ f$* and *$epi\ f_n$* for all $n \geqslant 1$. If $p_n \to p$ uniformly on the unit ball B of the product space $E \times R$, then $f_n \to f$ uniformly on each bounded set of E.

Proof. Since $f_n(0) \leqslant f(0) < 0$, p and $p_n (n \geqslant 1)$ are continuous Minkowski functionals on $E \times R$. Suppose, to the contrary, that there exist a bounded subset A of E and a subsequence of $\{f_n\}$ which is still denoted by $\{f_n\}$ such that $\{f_n\}$ fails to converge to f uniformly on A. Then there exist $\epsilon > 0$ and a sequence $\{x_n\} \subset A$ such that $|f(x_n) - f_n(x_n)| \geqslant \epsilon$ for all $n \geqslant 1$. Thus $f(x_n) - f_n(x_n) \geqslant \epsilon$ for all n since $f_n \leqslant f$. We observe that $(x_n, f_n(x_n)) \notin epi f$ for all $n \geqslant 1$, and these say $k_n \equiv p(x_n, f_n(x_n)) > 1$ for all $n \geqslant 1$. Note $p(x, r) = 1$ if and only if $r = f(x)$. We have $k_n^{-1}(x_n, f_n(x_n)) \in epi\ f$ with $f(k_n^{-1} x_n) = k_n^{-1} f_n(x_n)$.

Since $f(x_n) - f_n(x_n) \geqslant \epsilon$ for all n, we see that $\{k_n\}$ fails to converge to 1 and further that $p(x_n, f_n(x_n)) - p(x_n, f(x_n)) = k_n - 1$ fails to converge to 0.

On the other hand, by the hypothesis and by Lemma 4.4, $L_E(f_n) \leqslant L_E(f) < \infty$ for all $n \geqslant 1$. Therefore $\{f_n(x_n)\}$ is a bounded sequence and bounded by $L_E(f) \cdot M$, where $M = sup\{\|x\|: x \in A\}$, and further $\{(x_n, f_n(x_n))\}$ is bounded in $E \times R$. Uniform convergence of $\{p_n\}$ to p implies

$$k_n - 1 = p(x_n, f_n(x_n)) - 1 = p(x_n, f_n(x_n)) - p_n(x_n, f_n(x_n)) \to 0,$$

this contradicts that $k_n - 1$ fails to converge to 0.

THEOREM 4.6. *Suppose that E admits an equivalent locally uniformly convex norm. Then for every* w^**-lower semicontinuous b-Lipschitzian convex function f on* E^* *there exists a* w^**-lower semicontinuous generically Fréchet differentiable Lipschitzian convex function sequence* $\{f_n\}$ *with* $f_n \leqslant f_{n+1} \leqslant f$ *for all* $n \geqslant 1$ *such that* $f_n \to f$ *uniformly on each bounded subset of E.*

Proof. Suppose that f is a w^*-lower semicontinuous b-Lipschitzian convex function on E^*. Without loss of generality we assume $f(0) = -1$. Let p be the Minkowski functional on $E^* \times R$ generated by $epi\ f$. Then p is w^*-lower semicontinuous and norm continuous Minkowski functional since $epi\ f$ is w^*-closed convex and with the origin $(0, 0) \in int\ epi\ f$. Due to Theorem 3.3, there exist w^*-lower semicontinuous and norm continuous generically Fréchet differentiable Minkowski functionals $p_n (n = 1, 2, \ldots)$ with $p_n \leqslant p_{n+1} \leqslant p$ for all $n \geqslant 1$ such that $p_n \to p$ uniformly on each bounded subset of $E^* \times R$.

Let

$$C_n^* = \{(x^*, r) \in E^* \times R : p_n(x^*, r) \leqslant 1\}, \qquad \text{for all} \quad n \geqslant 1.$$

Then C_n^* is w^*-closed convex with $C_n^* \supset C_{n+1}^* \supset C^* \equiv epi\ f$ for each $n \geqslant 1$. We define f_n on E by

$$f_n(x^*) = inf\{r: (x^*, r) \in C_n^*\}, \qquad x^* \in E^*.$$

This definition is meaningful since $C_n^* \supset C^* \equiv epi\, f$, which imply that for every $x^* \in E^*$ and every n there is at least one $r \in [-\infty, \infty)$ with $r \leqslant f(x^*)$ such that $(x^*, r) \in C_n^*$. All f_n are w^*-lower semicontinuous and convex on E^* with $f_n \leqslant f_{n+1} \leqslant f$ since C_n^* are w^*-closed convex and with $C_n^* \supset C_{n+1}^* \supset C^*$ for all $n \geqslant 1$.

We claim that f_n^* are nowhere $-\infty$ valued on E^* for all sufficiently large $n \geqslant 1$.

Noting $f(0) = -1$, we have $p(0, -1) = 1$. Thus there exists $m > 0$ such that $1 \geqslant p_n(0, -1) \geqslant \frac{1}{2}$ for all $n \geqslant m$. Suppose, to the contrary, that there exists $x^* \in E^*$ such that $f_n(x^*) = -\infty$ for some $n \geqslant m$. By definition, $p_n(x^*, r) \leqslant 1$ for all r in R, and further $p_n(x^*/|r|, -1) \leqslant 1/|r|$ for all $r < 0$. Letting $r \to -\infty$, we see $(\frac{1}{2} \leqslant)\ p_n(0, -1) \to 0$, this is a contradiction.

We have shown that f_n are w^*-lower semicontinuous real-valued (hence, norm continuous), and p_n are exactly the Minkowski functionals generated by f_n for all sufficiently large $n \geqslant 1$. Applying Proposition 2.6 we know that f_n are generically Fréchet differentiable in E^*. It follows from Lemma 4.5 that $f_n \to f$ uniformly on each bounded subset of E^*.

Since f is b-Lipschitzian , f_n are also b-Lipschitzian for all sufficiently large $n \geqslant 1$ by Corollary 4.3. It remains to show each such f_n can be chosen to be Lipschitzian.

Case (I). f is Lipschitzian. We complete our proof by Lemma 4.4 since $f \geqslant f_n$ for all n.

Case (II). f is b-Lipschitzian. Let $L_n(f)$ denote the Lipschitz norm of f on $B_n^* \equiv \{x^* \in E^* : \|x^*\| \leqslant n\}$. By a simple argument of the inf-convolution sequence $\{f \,\Box\, n \,\|\cdot\|\}$ of f and $n \,\|\cdot\|$ on E^* defined for all $n \geqslant 1$ by

$$(f \,\Box\, n \,\|\cdot\|)\, x^* = inf\{f(y^*) + n\, \|x^* - y^*\| : y^* \in E^*\}, \qquad x^* \in E^*,$$

where $\|\cdot\|$ denotes the dual norm on E^*. We can find a sequence (a subsequence of $\{f \,\Box\, n \,\|\cdot\|\}$, to be exact) $\{g_n\}$ of w^*-lower semicontinuous Lipschitzian convex functions on E^* such that $g_n \leqslant g_{n+1} \leqslant f$ and $g_n = f$ in B_n^* for all $n \geqslant 1$.

Let $|\cdot|$ be an equivalent locally uniformly convex norm on $E \times R$, and let $D_n^* \equiv epi\, g_n$ and $D^* \equiv epi\, f$. We denote by D_n and D the polars of D_n^* and D^*, respectively, and by q_n and q the (extended-real-valued and lower semicontinuous) Minkowski functionals generated by D_n and D for all $n \geqslant 1$, resp.; and we also denote by q_n^* and q^* the Minkowski functionals on $E^* \times R$ generated by D_n^* and D^*, resp. Then we have for all $n \geqslant 1$

$$D_n^* \supset D_{n+1}^* \supset D^*, \qquad D_n \subset D_{n+1} \subset D,$$

$$q_n^* \leqslant q_{n+1}^* \leqslant q^*, \qquad q_n \geqslant q_{n+1} \geqslant q,$$

$$q_n^* = \sigma_{D_n}, \qquad q^* = \sigma_D, \qquad q_n = \sigma_{D_n^*}; \qquad \text{and} \qquad q = \sigma_{D^*}.$$

For all $n, k \in N$, let

$$D_{n,k} = \left\{ z \in E \times R : \frac{1}{k} |z| + q_n(z) \leqslant 1 \right\}.$$

Then $D_{n,k} \subset D_{n,k+1} \subset D_n$, and $D_{n,k} \subset D_{n+1,k}$ for all $n, k \in N$.

Clearly, D, D_n, and $D_{n,k}$ are closed convex bounded sets in $E \times R$ since $\sigma_D = q^*$ is continuous. Let $q^*_{n,k} = \sigma_{D_{n,k}}$. Then by Lemma 3.1, $q^*_{n,k}$ are generically Fréchet differentiable in $E^* \times R$ for all $n, k \in N$. We can easily check

(i) $q^*_{n,k} \leqslant q^*_{n,k+1} \leqslant q^*_n \leqslant q^*_{n+1} \leqslant q^*$ for all $n, k \in N$;
(ii) $q^*_{n,k} \to q^*_n$ uniformly on each bounded subset of $E^* \times R$;
(iii) $q^*_n \to q^*$ uniformly on each bounded subset of $E^* \times R$.

Hence $\{q^*_{n,n}\}$ is a monotone nondecreasing sequence and converges to q^* uniformly on each bounded subset of $E^* \times R$.

For each $n \geqslant 1$, let $h_n(x^*) = inf\{r: (x^*, r) \in D^*_{n,n}\}$, where $D^*_{n,n}$ denotes the polar of $D_{n,n}$. Then h_n are w^*-lower semicontinuous real-valued convex and with $epi\, h_n = D^*_{n,n}$ for all sufficiently large $n \geqslant 1$. We claim the sequence $\{h_n\}$ has the desired properties. Again by Lemma 2.2, the h_n are generically Fréchet differentiable for all sufficiently large $n \geqslant 1$ since $q^*_{n,n}$ are generated by $epi\, h_n$ and are generically Fréchet differentiable. Appling Lemma 4.5 we see $h_n \nearrow f$ uniformly on each bounded subset of E^*. Finally, by Lemma 4.4, h_n are Lipschitzian since $h_n \leqslant g_n$ and g_n are Lipschitzian on E^*. Hence, the proof is finished.

COROLLARY 4.7. *Every w^*-lower semicontinuous real-valued convex function can be approximated uniformly on every bounded set of the dual E^* of a separable space E by a w^*-lower semicontinuous Lipschitzian generically Fréchet differentiable convex function sequence.*

5. FINAL REMARKS

Combining Proposition 2.7 with Theorem 4.6 we immediately have

THEOREM 5.1. *Suppose that a Banach space E admits an equivalent locally uniformly convex norm, in particular, E is separable. Then for every extended-real-valued w^*-lower semicontinuous proper convex function f there exists a sequence $\{f_n\}$ of convex functions such that for all $n \geqslant 1$*

(i) $f_n \leqslant f_{n+1} \leqslant f$;

(ii) f_n *are* w^**-lower semicontinuous and Lipschitzian on* E^*;

(iii) f_n *are generically Fréchet differentiable in* E^*;

(iv) $\lim_{n\to\infty} f_n(x^*) = f(x^*)$ *for all* $x^* \in E^*$.

Remark 5.2. We know a locally uniformly convexifiable Banach space (i.e., a space admitting an equivalent locally uniformly convex norm), in particular, a separable space, would fail to have the Radon–Nikodým property. But combining Collier's theorem [5] with the main results of this paper we see that a locally uniformly convexifiable space is very close to the RNP. And yet, we could not substitute "generic Fréchet differentiability" for "generic Fréchet differentiability approximating."

Remark 5.3. Without additional assumptions, we could not substitute "everywhere Fréchet differentiable" for "generically Fréchet differentiable," since E^* admits an equivalent everywhere Fréchet differentiable off the origin and w^*-lower semicontinuous norm (i.e., a dual norm) if and only if E is reflexive. By the same methods presented in this paper we can show that "generic Fréchet differentiability" can be replaced by "everywhere (off the origin) Fréchet differentiability" in this paper if and only if E is reflexive.

Remark 5.4. Comparing with Mazur's (1933) theorem [14] "every continuous convex function on a separable Banach space is Gateaux differentiable at each point of a dense G_δ subset (i.e., generically Gateaux differentiable)," the version of this paper "every w^*-lower semicontinuous (real-valued) convex function on the dual of a separable space can be approximated by a generically Fréchet differentiable convex function sequence" is somewhat like a dual version of Mazur's theorem.

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